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ADDED MASS FOR FLUID–STRUCTURE VIBRATION PROBLEMS*

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SUMMARY

The problem of computing the vibration modes of a structure vibrating in a fluid is examined with specific application to ship hulls. In particular, methods which take proper account of the three-dimensional nature of the water movements are described. Fluid singularities involving either a line doublet on the intersection of the water surface with the plane of symmetry or distributed sources on retracted boundaries are particularly effective at modelling the fluid flow and appear to give better numerical efficiencies than finite element or boundary element methods. The effects of the extra coefficients in the mass matrix arising from added mass on the various methods of eigensolution are discussed.

INTRODUCTION

From the early 1960s the capacity of digital computers has enabled engineers to carry out stress analyses of complicated structures subject to static loadings. The method now generally adopted is the finite element method because of the ease with which it can cope with problems whatever are their geometrical forms and also the way more accurate results may be obtained when required by using a finer mesh in the idealization. Although many thousands of variables may be used for the analysis of large structures such as aircraft, ships, bridges etc., the solution of the stiffness equations is not too cumbersome because advantage may be taken of sparsity.

The eigenvalue problem most often encountered with such complicated structures is that of finding their lower frequencies of vibration. In this case the accelerations of the structure at any particular time yield inertia forces which can be deemed to balance the structural forces giving equations of the form

$$\mathbf{M}\ddot{\chi} + \mathbf{K}\chi = \mathbf{0}, \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are $n \times n$ symmetric mass and stiffness matrices and χ and $\ddot{\chi}$ are vectors of displacements and accelerations, respectively. Making use of the known sinusoidal nature of the vibration it is possible to substitute

$$\chi = \mathbf{x} \sin(\omega t + \varepsilon), \tag{2}$$

where ω is the frequency of vibration. Equation (1) then yields the linear generalized eigenvalue form

$$\mathbf{M}\mathbf{x} = \omega^2 \mathbf{K}\mathbf{x}. \tag{3}$$

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Where the mass entirely derives from the self-weight of the structure or from material directly attached to the structure which moves with it, the mass matrix will have the same sparsity structure as **K**. The techniques which are most widely used for solving these types of equation are simultaneous and subspace iteration,[1-3] the Lanczos method[4,5] and Sturm sequence methods.[3,6,7] These all involve factorization of the stiffness matrix or matrices of the form $\mathbf{K} - \mu\mathbf{M}$ and hence the amount of fill-in during factorization of **K** or $\mathbf{K}-\mu\mathbf{M}$ is an important parameter in their efficiency. Such analyses have generally been applied to aircraft, bridges and other types of structure by neglecting the mass of the surrounding air. However for structures surrounded or partially surrounded by water such as ships, submarines and dams, motion of the water in the neighbourhood of the structure must be taken into account in the analysis. It is this aspect, with particular reference to ship hulls, that is the consideration of this paper.

EARLY TREATMENT OF FLUID INTERACTION

Although Lamb[8] investigated the accelerated motion of a submerged cylinder, the added mass effect for ship vibration was only properly recognized from the experimental work of Nicholls[9] in 1924 and the mathematical analysis using conformal transformations of Lewis[10] in 1929. The method of analysis for ship vibration which developed from this work was based on the strip theory assumption that the water motion in the region of a particular cross-section of the ship corresponds to that of an infinitely long ship of the same cross-section going through the same motion. Assuming that the water is incompressible, for a vertical vibration of the ship there will be water movement between the hull and the free surface on the downbeat (Figure 1(a)) and back again on the upbeat. The added mass effect comes from the pressures transmitted to the hull arising from the inertia of the water. By including a reflected image of the hull in the free surface (Figure 1(b)) it is possible to solve instead the problem of a vibrating body completely submerged in fluid. A simple case occurs where the hull below the waterline is semicircular, in which case the added mass is exactly equal to the weight of the water displaced by the hull. Since the total weight of the ship must equal the total weight of the displaced water, the added mass is likely to be of the same order of magnitude as the actual mass of the ship and is therefore of considerable importance.

However it was found necessary to include correction factors to take account of the fact that the water motion is really three-dimensional. For instance in the heave mode there will be some

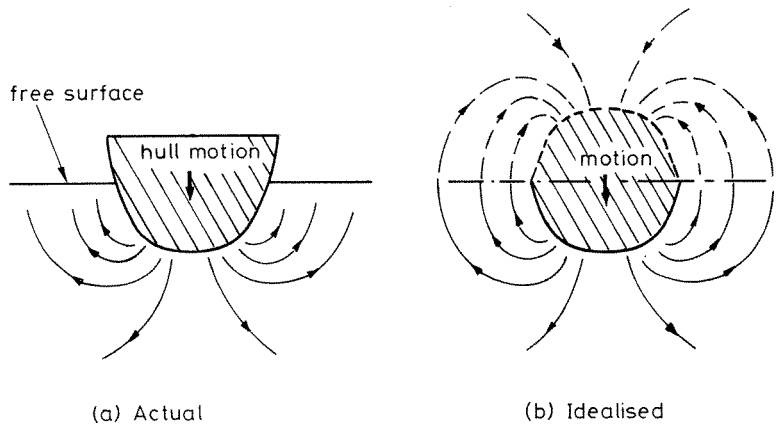

Figure 1. Water movement in ship hull vibration according to strip theory

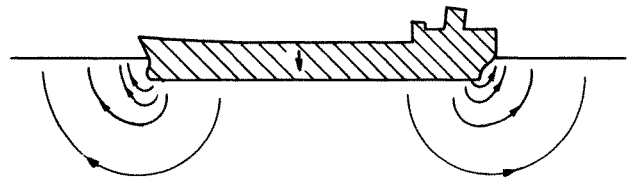

Figure 2. Longitudinal water movement in the plane of symmetry for the heave mode of a ship hull

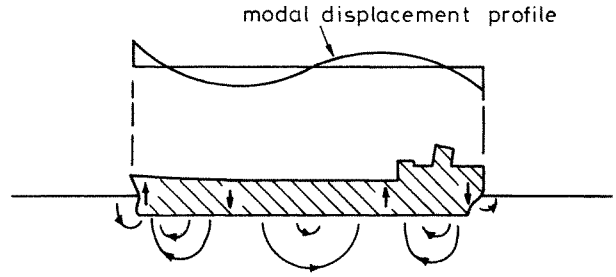

Figure 3. Longitudinal water movement in the plane of symmetry for the second bending mode of a ship hull

longitudinal water movement as illustrated in Figure 2. For ships of narrow beam this gives a slight reduction in the added mass effect and a larger reduction for ships of broad beam. Lewis corrected strip theory added mass by means of a factor evaluated from the motion of water around an ellipsoid of revolution having the same length and breadth as the ship. However, since higher order vibration modes exhibit nodes along the length of the hull, longitudinal water movements are facilitated by the shorter flow paths involved (Figure 3). Hence corrections required to the strip theory results are larger for increasing mode number. From experimental investigations of several authors, Townsin[11] proposed the simple correction factor

$$J_n = 1{\cdot}02 - \frac{3B}{L}\left(1{\cdot}2 - \frac{1}{n}\right),$$

where B and L are the beam and length of the ship, respectively, and n is the mode number.

Such methods can be criticized because they avoid modelling the three-dimensional movement of water around the actual hull of the ship and therefore have in-built errors which cannot be reduced by refinement. Also, because the correction factors are frequency dependent, the computational of each vibration frequency of the ship hull needs to be carried out individually. Thus, not only are transformation methods ruled out, but also the powerful vector iterative methods of simultaneous iteration, subspace iteration and Lanczos iteration. Furthermore the change of mass from mode to mode destroys the orthogonality condition between the different modes of vibration.

THREE-DIMENSIONAL FLUID EQUATIONS

If the velocity of an inviscid fluid at the point (x, y, z) has the components u, v and w in the

(orthogonal) x, y and z directions, the Eulerian equations of motion give

$$\left.\begin{aligned} -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} &= \frac{\partial u}{\partial t} + \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right), \\ -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}y} &= \frac{\partial v}{\partial t} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right), \\ -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}z} &= \frac{\partial w}{\partial t} + \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right), \end{aligned}\right\} \tag{4}$$

where ρ is density, t is time, g is the gravitational constant and p is the pressure due to motion (i.e. not including the static pressure). Furthermore, assuming that the fluid is incompressible

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{5}$$

If the vibration motion is small the convective accelerations (the bracketed terms in the Eulerian equations) can be neglected. Hence equations (4) and (5) yield the Laplace equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0. \tag{6}$$

The boundary condition at the surface of the hull is

$$\frac{\partial p}{\partial n} = -\rho\frac{\partial V_n}{\partial t} = -\rho a_n, \tag{7}$$

where n is the direction normal to the surface and V_n and a_n are the normal velocity and acceleration, respectively.

FINITE ELEMENT IDEALIZATION FOR A COMPLETE SYSTEM

One way of analysing such a system is to develop a finite element discretization which involves fluid as well as structural elements.[12] For instance Armand and Orsero[13] use such an idealization to analyse the vibration characteristics of a simplified barge structure. By using two planes of symmetry, only one quarter of the barge was analysed. The structure was modelled using 660 finite elements involving 1340 displacement variables at 476 nodes. The fluid domain was modelled using 200 twenty-node three-dimensional elements involving over 900 fluid nodes (Figure 4). The extent of the water domain was 150 m longitudinally, 60 m transversely and 80 m vertically compared with 90 m, 20 m and 15 m for the corresponding dimensions of the quarter barge. The free surface and fluid external boundary conditions were $p = 0$ and, on the plane of symmetry, $\partial p/\partial n = 0$.

The discretization of Laplace's equation for the fluid yields equations of the form

$$\mathbf{K}_{\mathrm{F}}\mathbf{p}(t) = -\rho\mathbf{L}\ddot{\mathbf{u}}(t), \tag{8}$$

where $\mathbf{p}(t)$ is a vector of dynamic pressures at the fluid nodes, $\ddot{\mathbf{u}}(t)$ is a vector of accelerations at the hull-fluid boundary and ρ is the density of the fluid. The corresponding added mass matrix describing the pressure forces at the hull-fluid boundary is given by

$$\mathbf{M}_{\mathrm{A}} = \rho\mathbf{L}^{\mathrm{T}}\mathbf{K}_{\mathrm{f}}^{-1}\mathbf{L}. \tag{9}$$

The resulting equations of undamped, unforced vibration of the system are specified as

$$(\mathbf{M}_{\mathrm{S}} + \mathbf{M}_{\mathrm{A}})\ddot{\mathbf{u}}(t) + \mathbf{K}_{\mathrm{S}}\mathbf{u}(t) = \mathbf{0}, \tag{10}$$

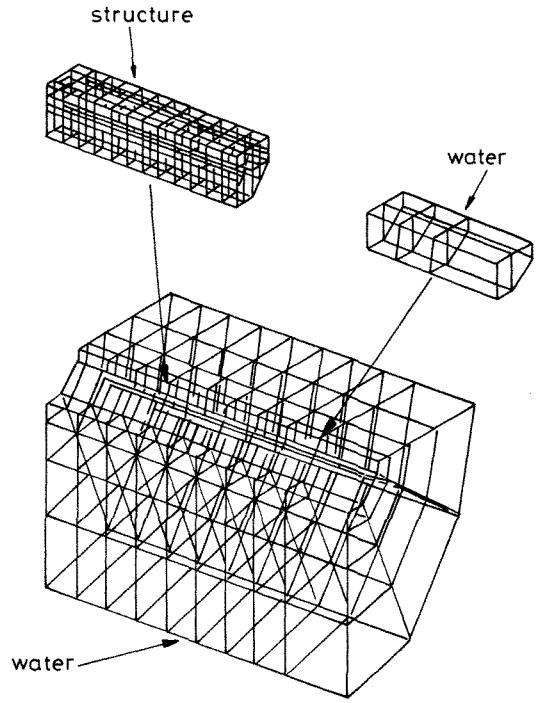

Figure 4. Armand and Orsero finite element idealization

where $\mathbf{M}_S$ and $\mathbf{K}_S$ are the structural mass and stiffness matrices, respectively.

Analysis of ships by this method would normally require more variables because of lack of fore and aft symmetry and also the specification of fluid elements would be complicated by the non-uniformity of the ship cross-section. A further problem with adopting this method is that it is necessary to know how far to extend the fluid domain in order to ensure that the external fluid boundary constraints are negligible, although this problem may be partially alleviated by the use of infinite elements.[14–15]

USE OF FLUID SINGULARITIES IN AXISYMMETRIC PROBLEMS

Alternative methods of representing the motion of the fluid which may be used for three-dimensional analyses involve the use of fluid singularities either inside or at the surface of the structure.

One of the simplest possible methods applicable to vertical vibration of ships is to assume that the hull has the geometry of a cylinder of varying diameter with its axis in the plane of the surface. Steady flow round the cylinder can then be idealized by means of a distribution of doublets with their axes vertical and lying along the axis of the cylinder as shown in Figure 5. A doublet of strength $\hat{\mu}$ acting at point Q gives rise to a potential at point P (Figure 6) of

$$\phi = \frac{\cos\hat{\theta}}{4\pi\hat{r}^2}\hat{\mu}, \tag{11}$$

where $\hat{r}$ is the distance PQ and $\hat{\theta}$ is the angle between PQ and the axis of the doublet. With r as the

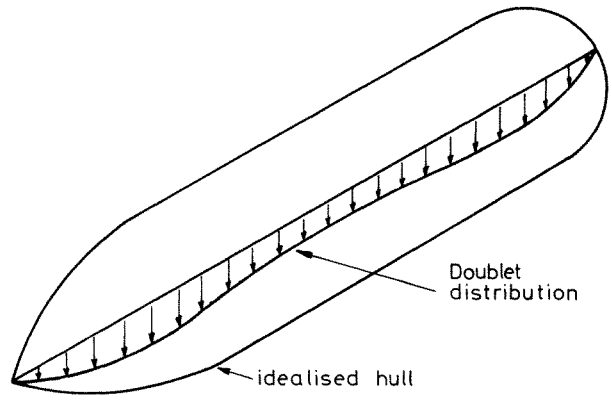

Figure 5. A simple idealization for hull vibration

local radius at P and θ the corresponding angle to the vertical

$$\hat{r}\cos\hat{\theta} = r\cos\theta. \tag{12}$$

Hence

$$\phi = \frac{r\cos\theta}{4\pi[r^2 + (z-\hat{z})^2]^{3/2}}\hat{\mu} \tag{13}$$

where z and $\hat{z}$ are the longitudinal co-ordinates of P and Q, respectively. The local fluid velocity normal to the surface is given by

$$\frac{\partial\phi}{\partial r} = \frac{[(z-\hat{z})^2 - 2r^2]\cos\theta}{4\pi[r^2 + (z-\hat{z})^2]^{5/2}}\hat{\mu} \tag{14}$$

and the normal acceleration at P due to the complete doublet distribution is given by integration

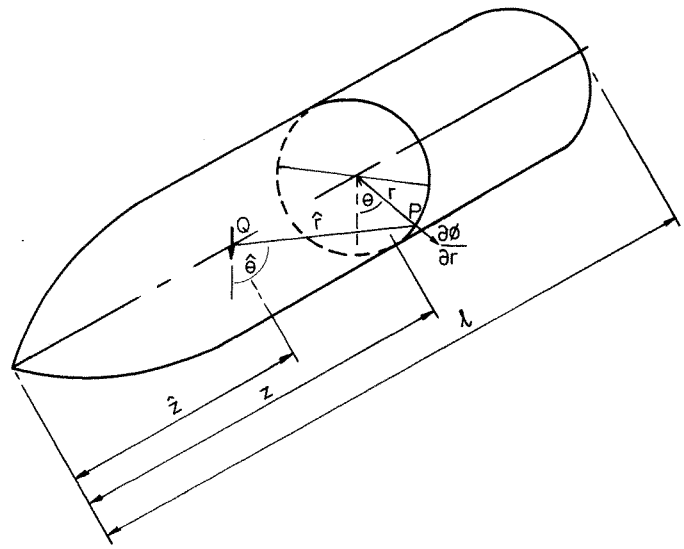

Figure 6. Normal velocity due to doublet increment

using

$$\hat{\mu} = \mu \mathrm{d}\hat{z}, \tag{15}$$

giving

$$\frac{\partial \dot{\phi}}{\partial r} = \int_0^l \frac{[(z-\hat{z})^2 - 2r^2]\cos\theta}{4\pi[r^2 + (z-\hat{z})^2]^{5/2}} \dot{\mu} \mathrm{d}\hat{z}. \tag{16}$$

However, if the cross-section containing P has a vertical acceleration of $\ddot{x}$, the normal acceleration at P is given by $\ddot{x}\cos\theta$. Hence

$$\ddot{x} = \int_0^l \frac{[(z-\hat{z})^2 - 2r^2]}{4\pi[r^2 + (z-\hat{z})^2]^{5/2}} \dot{\mu} \mathrm{d}\hat{z} \tag{17}$$

For numerical modelling, the hull is divided into n segments with the acceleration of the centre cross-section of each segment being matched with the acceleration of the structure. In order to do this with n variables describing the longitudinal strength distribution of the doublets and $\ddot{\mathbf{x}}$ as a vector of n segment accelerations,

$$\ddot{\mathbf{x}} = \mathbf{C}\dot{\boldsymbol{\mu}}. \tag{18}$$

If $\boldsymbol{\mu}$ describes a set of discrete doublets, each coefficient of $\mathbf{C}$ is obtained directly from equation (17) by removing the integral sign and substituting the appropriate values of r, z and $\hat{z}$. In other cases, such as for the stepped doublet distribution shown in Figure 7, an integration is involved for each coefficient.

From equations (4), considering that $u = \partial\phi/\partial x$, it follows that

$$p = -\rho\dot{\phi}. \tag{19}$$

The vertical component of force/unit length exerted on the lower semi-circle of a longitudinal segment of the hull is given by

$$f = -\int_{-\pi/2}^{\pi/2} pr\cos\theta \mathrm{d}\theta = \rho r \int_{-\pi/2}^{\pi/2} \dot{\phi}\cos\theta \mathrm{d}\theta. \tag{20}$$

Using equations (13) and (15) and integrating the total force per unit length due to the complete doublet distribution

$$f = \frac{\rho r^2}{4\pi}\int_{-\pi/2}^{\pi/2} \cos^2\theta \mathrm{d}\theta \int_0^l \frac{\dot{\mu}\mathrm{d}\hat{z}}{[r^2 + (z-\hat{z})^2]^{3/2}} = \frac{\rho r^2}{8}\int_0^l \frac{\dot{\mu}\mathrm{d}\hat{z}}{[r^2 + (z-\hat{z})^2]^{3/2}}. \tag{21}$$

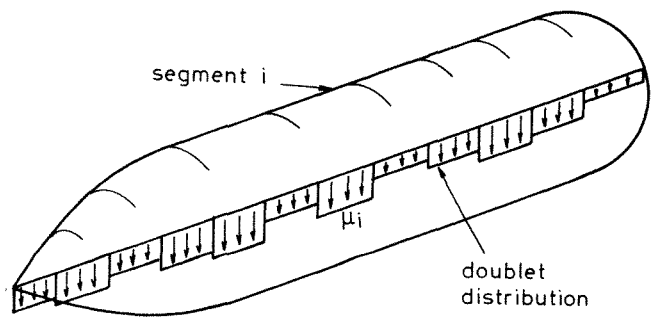

Figure 7. Stepped doublet distribution

Integration of the force for each segment in terms of each doublet variable gives the coefficients of a matrix **Q** such that

$$\mathbf{f} = \mathbf{Q}\dot{\boldsymbol{\mu}}, \tag{22}$$

where **f** is the vector of segment forces. Eliminating $\dot{\boldsymbol{\mu}}$ between equations (17) and (22) gives the added mass matrix for the fluid as

$$\mathbf{M}_A = \mathbf{Q}\mathbf{C}^{-1}. \tag{23}$$

Comparison have been made for the case of circular cylinders with results by Kumai[16] using energy methods and for the case of ellipsoids of revolution with results given by Hicks.[17] In both cases agreement is good.[18]

It is possible to use this method to model the vibration characteristics of ships which are not semi-circular in cross-section by replacing each ship segment with the semi-circular cross-section which exhibits the same two-dimensional virtual mass as the actual cross-section.

USE OF FLUID SINGULARITIES IN MORE GENERAL THREE-DIMENSIONAL PROBLEMS

In order to obtain idealizations which model the fluid accelerations more accurately it is necessary to replace the doublets on the central axis by a number of fluid singularities closer to or on the surface of the ship. Agar[19] has investigated the use of line singularities of the form shown in Figure 8 for ship hull vibration. Both doublets and simple sources were investigated and simple sources were found to be the most appropriate. The cross-section was idealized as a series of segments with the centre-point of each segment being a control point at which the equation for normal velocity is satisfied. A source is then placed on the bisector of each segment and the free surface boundary condition maintained by placing a sink of equal strength in the mirror image position (Figure 9). Best results were obtained when the sources were positioned at a distance from the control point approximately equal to the length of the corresponding segment. If the sources are much closer to the hull surface, erroneous results will be obtained because fluid will not be prevented from leaking through the hull surface between the control points in the simulation. Alternatively if the singularities are placed much further away from the hull surface, the equations may become sufficiently ill-conditioned for the accuracy of the results to be affected.

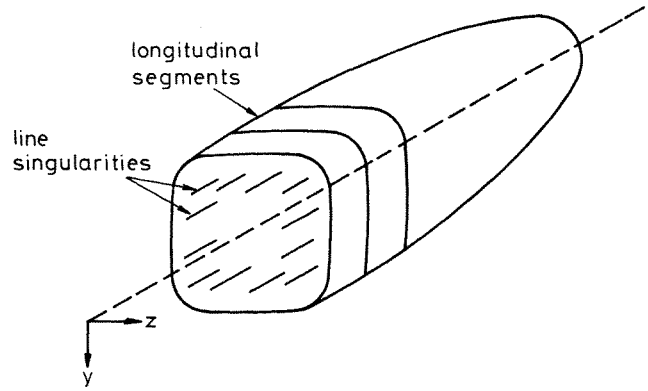

Figure 8. Agar singularity distribution

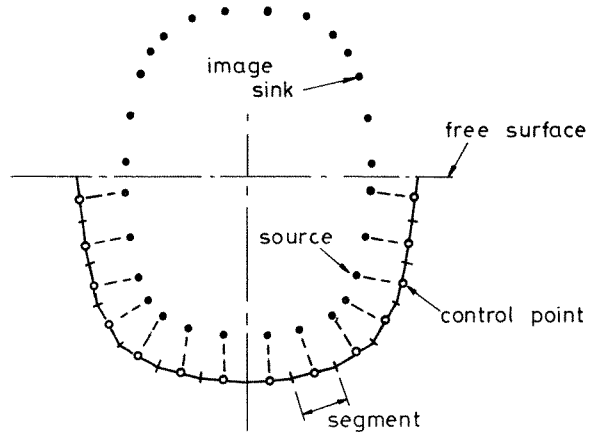

Figure 9. Arrangement of cross-section with Agar sources

McHenry[20] has investigated several arrangements for singularity distributions to represent two-dimensional flows around bodies of arbitrary shape. One such scheme he investigated used uniformly distributed line sources placed on a retracted boundary as shown in Figure 10. The position of the retracted boundary is determined uniquely from assigned surface segment positions and a retraction parameter f. Typically D lies on the bisector of angle ACE with

$$\frac{1}{CD} = \frac{1}{2f}\left(\frac{1}{AC} + \frac{1}{CE}\right). \tag{24}$$

When the segments are short the retracted boundary comes closer to the actual boundary and, furthermore, when a short segment is placed next to a long segment the amount of retraction will be influenced more by the short segment than the long segment. McHenry found that $f \simeq 0{\cdot}5$ was effective and that two-dimensional flow was modelled better than if the cruder Agar singularities were used. He was able to show that the modelling of the surface flows was much more accurate for retracted boundaries than it was for the case where $f = 0$ which corresponds with more common

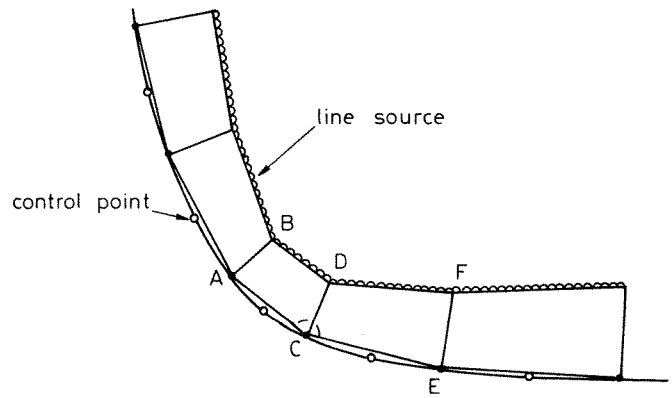

Figure 10. Arrangement of cross-section with retracted line sources having retraction parameter $f = 0{\cdot}5$

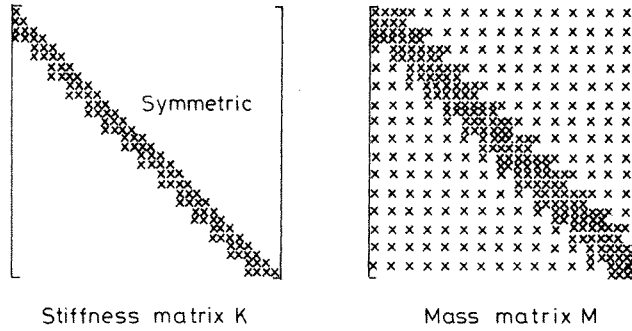

Figure 11. Sparsity structure of equations for a simple beam idealization of a ship hull using fifteen segments

applications of the boundary element technique. One sequence of tests showed that the modelling of surface velocities was as good, using retracted boundaries, as when three times as many surface elements were employed. The use of retracted boundaries has also been advocated by Pien[21] and, for other types of field problems, by Eskridge,[22] Oliviera[23] and Singer *et al.*[24]

It is therefore apparent that techniques exist for modelling the fluid motion so that the boundary conditions conform closely to those at the actual hull profile provided that extra variables are used in the fluid idealization.

THE COMBINED EQUATIONS OF MOTION FOR THE SIMPLEST CASE

The simplest treatment of ship vibration using three-dimensional fluid singularities is by combining the axisymmetric idealization for fluid flow with a simple beam theory representation of the structure. Provided that the hull profile and mass distribution are both symmetric about the ship centre plane, the equations for vertical vibration take the form shown in Figure 11 in which the variables are $\{x_1, \theta_1, x_2, \theta_2, \ldots\}$ where $x_1, x_2, \ldots$ are vertical displacements at the centres of each segment and $\theta_1, \theta_2, \ldots$ are the corresponding rotations, the segments being numbered consecutively. The mass matrix has a symmetric banded contribution arising from the ship structure and an unsymmetric contribution which couples all the displacement variables arising from the added mass of the fluid.

One problem arises because of the lack of symmetry in $\mathbf{M}$. Since the equations model a conservative system and complex eigenvalues have no physical meaning, the need to use unsymmetric eigensolution procedures appears to be an unnecessary embarrassment and indeed it has been found that replacing the added mass matrix $\mathbf{M}_A$ by the symmetric equivalent $\frac{1}{2}(\mathbf{M}_A + \mathbf{M}_A^T)$ makes little difference to the result.[25]

EQUATIONS WITH IMPROVED FLUID MODELLING

If the modelling technique of Agar is adopted or the method of McHenry is extended to three dimensional configurations each segment of the hull will be allocated several control points and source variables. If there are a total of m control points and source variables the following matrix relationships can be defined:

$$\left.\begin{aligned} \ddot{\mathbf{w}} &= \mathbf{C}\dot{\boldsymbol{\mu}}, \\ \mathbf{p} &= \mathbf{D}\dot{\boldsymbol{\mu}}, \end{aligned}\right\} \tag{25}$$

where $\ddot{\mathbf{w}}$ is the vector of local normal accelerations at the control points and $\mathbf{p}$ is a vector of integrated pressures acting on the panels. Both $\mathbf{C}$ and $\mathbf{D}$ are fully populated matrices of order $m \times m$. Thus

$$\mathbf{p} = \mathbf{D}\mathbf{C}^{-1}\ddot{\mathbf{w}}. \tag{26}$$

The accelerations at the control points may be obtained in terms of the segment accelerations giving

$$\ddot{\mathbf{w}} = \mathbf{Y}\ddot{\mathbf{x}}, \tag{27}$$

where $\mathbf{Y}$ is of order $m \times n$ with only one non-zero element per row. The corresponding transformation giving the segment force in terms of the forces exerted on each of its component panels is

$$\mathbf{f} = \mathbf{Y}^{\mathrm{T}}\mathbf{p}. \tag{28}$$

Hence the added mass matrix is the product

$$\mathbf{M}_{\mathrm{A}} = \mathbf{Y}^{\mathrm{T}}\mathbf{D}\mathbf{C}^{-1}\mathbf{Y}, \tag{29}$$

having the form shown in Figure 12(a). Agar discovered that the off-diagonal blocks of the $\mathbf{C}$ matrix tended to be weak, particularly away from the diagonal and could be omitted without much effect on the result as shown in Figure 12(b). The best solution procedure in either case is to solve

$$\left.\begin{aligned}\mathbf{C}\mathbf{Z} &= \mathbf{Y},\\ \mathbf{X}^{\mathrm{T}} &= \mathbf{Y}^{\mathrm{T}}\mathbf{D},\end{aligned}\right\} \tag{30}$$

and multiply

making use in both cases of sparsity in $\mathbf{Y}$, then

$$\mathbf{M}_{\mathrm{A}} = \mathbf{X}^{\mathrm{T}}\mathbf{Z}. \tag{31}$$

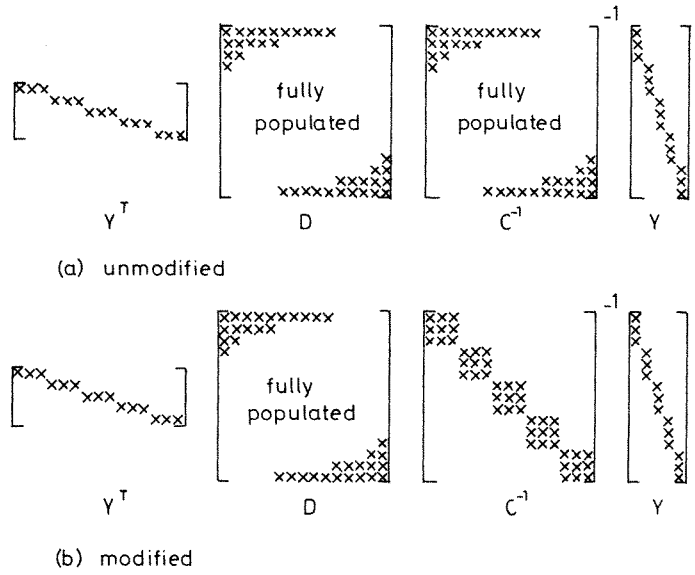

Figure 12. Matrix product giving added mass matrix (using 5 segments with 3 singularities per segment)

If there are n segments and r sources per segment the main term in the number of multiplications required to form $\mathbf{M}_A$ if $n \gg 1$, $r \gg 1$ is approximately $n^3r^3/3$ with $\mathbf{C}$ unmodified and n^2r^2 with $\mathbf{C}$ modified. This means that, for instance, with 12 segments and 6 sources/segment used for the barge analysis (Figure 4) approximately 7500 or 190,000 flops are required to form $\mathbf{M}_A$ depending on whether $\mathbf{C}$ is modified or not. This compares with about 2,000,000 flops to obtain the equivalent added mass matrix using the finite element idealization given in Figure 4 taking account of sparsity in the formulation. (With tridiagonal blocks in $\mathbf{C}$ retained the computing requirement is approximately 32,000 flops).

EIGENSOLUTION OF BEAM EQUATIONS

In the solution of equations in which the hull is modelled as a beam, the order of the eigensolution problem to be solved will only be twice the number of longitudinal strips adopted in the idealization. It is therefore possible to consider transformation methods, the simplest being a generalized Jacobi method which obtains the eigenvalues in a single iterative process.[3,26] Because body movements of the ship are unrestrained, the structural stiffness matrix is singular. If buoyancy effects are included then additional stiffness terms are present which render the stiffness matrix non-singular, and equation (3) can be transformed to the symmetric standard eigenvalue problem

$$\mathbf{L}^{-1}\mathbf{M}\mathbf{L}^{-\mathrm{T}}\mathbf{y} = \frac{1}{\omega^2}\mathbf{y}, \tag{32}$$

where $\mathbf{L}$ is the Choleski factor of $\mathbf{K}$ and $\mathbf{y} = \mathbf{L}^{\mathrm{T}}\mathbf{x}$. The eigenvalues of $\mathbf{L}^{-1}\mathbf{M}\mathbf{L}^{-\mathrm{T}}$ can then be found, for instance, by Householder tridiagonalization and the method of bisection. However, the buoyancy forces only arise when vertical vibrations of the ship are being investigated. When lateral or torsional vibrations of the ship take place the stiffness matrix will be singular unless small artificial buoyancy terms are added. One method of reducing the equations to standard form when $\mathbf{K}$ is singular[1] is to add a small component of the mass matrix to the stiffness matrix, thus

$$(\mathbf{K} + \alpha\mathbf{M})\mathbf{x} = (\omega^2 + \alpha)\mathbf{M}\mathbf{x}. \tag{33}$$

With α positive, the modified stiffness matrix may now be factorized. If $\mathbf{K} + \alpha\mathbf{M} = \tilde{\mathbf{L}}\tilde{\mathbf{L}}^{\mathrm{T}}$, the eigenvalues of $\tilde{\mathbf{L}}^{-1}\mathbf{M}\tilde{\mathbf{L}}^{-\mathrm{T}}$ are related to the frequencies through

$$\lambda = 1/(\omega^2 + \alpha) \tag{34}$$

This modification destroys the narrow bandwidth of the matrix to be factorized and so increases significantly the computing requirement.

EQUATIONS WITH IMPROVED STRUCTURAL MODELLING

Improved structural modelling can be obtained by replacing the beam formulation by either a two-dimensional[27] or three-dimensional finite element formulation. If this is done it is still possible to retain the same fluid modelling provided that distortion of cross-sections of the ship are neglected and the added mass for each cross-section is lumped at one point on the plane of symmetry. In this case the added mass only couples one variable of each cross-section. If the variables in the finite element formulation are ordered in sequence from bow to stern, as they would be in a frontal ordering scheme for efficient factorization of the structural stiffness matrix, then the mass matrix will take the form shown in Figure 13.

Because the equations are of higher order than when using a beam idealization it is essential that advantage is taken of sparsity structures in the equations and hence a form of simultaneous or

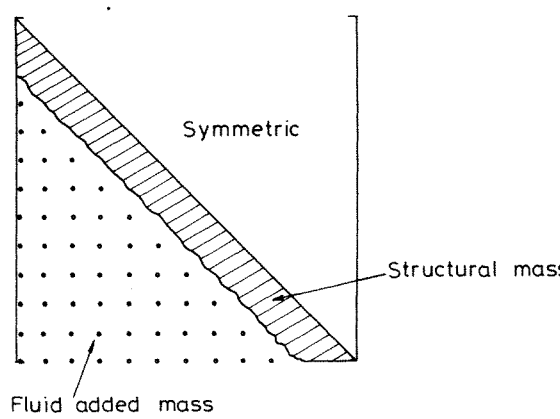

Figure 13. Sparsity structure of mass matrix for finite element structural idealization and simple fluid coupling

subspace iteration, the Lanczos method or the Sturm sequence method is likely to be used. In the case where **K** is non-singular and can be factorized application of standard simultaneous or subspace iteration or the Lanczos method to the unshifted equations avoids the need to factorize a matrix of the same structure as the mass matrix. However, in other cases such as the Sturm sequence method, the spectral Lanczos method[28] and some forms of accelerated simultaneous and subspace iteration methods,[29,30] the matrix to be factorized will have the same sparsity structure as **M**. If the sparsity structure shown in Figure 13 is retained, fill-in will arise in each of the long rows of $\tilde{\mathbf{L}}$ containing fluid coupling terms. Alternatively it is possible to rearrange the node ordering so that all the variables involving fluid coupling are ordered last. However whether this is done by a suitably modified nested dissection or frontal method it is likely to give a significantly less efficient solution than for the case where added mass is not present.

EQUATIONS WHICH ALLOW FOR FULL STRUCTURAL DISTORTION

If it is to be assumed that cross-sections distort, then the control point accelerations need to be defined in terms of the accelerations of the local structural nodes. This means that **Y** will tend to have at least as many columns as rows so that the added mass will couple displacements associated with structural nodes all round the surface of the hull. The mass matrix therefore becomes much less sparse than it is in Figure 13 and avoidance of factorizing the shifted matrix becomes even more rewarding. In cases where factorization of the shifted matrix is to be adopted, good accuracy should be obtained if the more accurate mass terms involving individual control point accelerations are used for coupling adjacent panels, reverting to the simple mass terms for coupling segments which are not adjacent to each other. This means that, if a frontal ordering scheme is used for the variables, most of the rows of the matrix to be factored will still retain a band structure.

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